

# ALL EXTENSIONS OF $C_2$ BY $C_{2^{n+1}} \times C_{2^{n+1}}$ ARE GOOD

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**ABSTRACT.** Let  $C_m$  be a cyclic group of order  $m$ . We prove that if the group  $G$  fits into an extension  $1 \rightarrow C_{2^{n+1}}^2 \rightarrow G \rightarrow C_2 \rightarrow 1$  then  $G$  is good in the sense of Hopkins-Kuhn-Ravenel, i.e.,  $K(s)^*(BG)$  is evenly generated by transfers of Euler classes of complex representations of subgroups of  $G$ . Previously this fact was known for  $n = 1$ .

**Keywords:** Morava  $K$ -theory, Transfer homomorphism, Euler class.

## 1. INTRODUCTION AND STATEMENTS

This paper is concerned with analyzing the 2-primary Morava  $K$ -theory of the classifying spaces  $BG$  of the groups in the title. In particular it answers the question whether transfers of Euler classes suffice to generate  $K(s)^*(BG)$ . Here  $K(s)$  denotes Morava  $K$ -theory at prime  $p = 2$  and natural  $s > 1$ . The coefficient ring  $K(s)^*(pt)$  is the Laurent polynomial ring in one variable,  $\mathbb{F}_2[v_s, v_s^{-1}]$ , where  $\mathbb{F}_2$  is the field of 2 elements and  $\deg(v_s) = -2(2^s - 1)$  [9]. So the coefficient ring is a graded field in the sense that all its graded modules are free, therefore Morava  $K$ -theories enjoy the Künneth isomorphism. In particular, we have for the cyclic group  $C_{2^{n+1}}$  that as a  $K(s)^*$ -algebra

$$K(s)^*(BC_{2^{n+1}}^2) = K(s)^*(BC_{2^{n+1}}) \otimes K(s)^*(BC_{2^{n+1}}),$$

whereas  $K(s)^*(BC_{2^m}) = K(s)^*[u]/(u^{2^{ms}})$ . So that

$$K(s)^*(BC_{2^{n+1}}^2) = K(s)^*[u, v]/(u^{2^{(n+1)s}}, v^{2^{(n+1)s}}),$$

where  $u$  and  $v$  are Euler classes of canonical complex linear representations.

The definition of good groups in the sense [8] is as follows.

a) For a finite group  $G$ , an element  $x \in K(s)^*(BG)$  is good if it is a transferred Euler class of a complex subrepresentation of  $G$ , i.e., a class of the form  $Tr^*(e(\rho))$ , where  $\rho$  is a complex representation of a subgroup  $H < G$ ,  $e(\rho) \in K(s)^*(BH)$  is its Euler class (i.e., its top Chern class, this being defined since  $K(s)^*$  is a complex oriented theory), and  $Tr : BG \rightarrow BH$  is the transfer map.

(b)  $G$  is called to be good if  $K(s)^*(BG)$  is spanned by good elements as a  $K(s)^*$ -module.

Recall not all finite groups are good as it was originally conjectured in [8]. For odd prime  $p$  a counterexample to the even degree was constructed In [11]. The problem to construct 2-primary counterexample conjecture remains open.

Our main result is as follows

**Theorem 1.1.** *All extensions of  $C_2$  by  $C_{2^{n+1}}^2$  are good.*

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*Key words and phrases.* Transfer, Morava  $K$ -theory.

For  $n = 1$  the statement of the theorem was known. See [2], [4], [13], [15] for detailed discussion and examples.

Of course the Serre spectral sequence is used throughout the paper. However, if to operate straightforward, even for  $s = 2$ ,  $n = 1$ , this requires a serious computational effort and use of computer, see [14] p.78. We simplify the task of calculation with invariants by suggesting the special bases for particular  $C_2$ -modules  $K(s)^*(BH)$ , see Lemma 4.1 and Lemma 4.2. This simple but comfortable idea is our key tool to prove Theorem 1.1. We will prove it for the semi-direct products

$$(1) \quad (C_{2^{n+1}} \times C_{2^{n+1}}) \rtimes C_2.$$

Then the general case follows because of the fact that the Serre spectral sequence does not see the difference between the semi-direct products and their non-split versions.

## 2. PRELIMINARIES

Recall [7] there exist exactly 17 non-isomorphic groups of order  $2^{2n+3}$ ,  $n \geq 2$ , which can be presented as a semidirect product (1). Each such group  $G$  is given by three generators  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  and the defining relations

$$\mathbf{a}^{2^{n+1}} = \mathbf{b}^{2^{n+1}} = \mathbf{c}^2 = 1, \mathbf{ab} = \mathbf{ba}, \mathbf{c}^{-1}\mathbf{ac} = \mathbf{a}^i\mathbf{b}^j, \mathbf{c}^{-1}\mathbf{bc} = \mathbf{a}^k\mathbf{b}^l$$

for some  $i, j, k, l \in \mathbb{Z}_{2^{n+1}}$  ( $\mathbb{Z}_{2^m}$  denotes the ring of residue classes modulo  $2^m$ ). In particular one has the following

**Proposition 2.1.** (see [7] ) *Let  $n$  be an integer such that  $n \geq 2$ . Then there exist exactly 17 non-isomorphic groups of order  $2^{2n+3}$  which can be presented as a semi-direct product (1). They are:*

$$\begin{aligned}
G_1 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b} \rangle, \\
G_2 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{1+2^n}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle, \\
G_3 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{ab}^{2^n}, \mathbf{cbc} = \mathbf{b} \rangle, \\
G_4 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{1+2^n} \mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle, \\
G_5 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle, \\
G_6 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1+2^n}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle, \\
G_7 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1} \mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle, \\
G_8 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1+2^n} \mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle, \\
G_9 &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{ab}^{2^n}, \mathbf{cbc} = \mathbf{a}^{2^n} \mathbf{b}^{1+2^n} \rangle, \\
G_{10} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle, \\
G_{11} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1} \mathbf{b}^{2^n}, \mathbf{cbc} = \mathbf{a}^{2^n} \mathbf{b}^{-1+2^n} \rangle, \\
G_{12} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle, \\
G_{13} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle, \\
G_{14} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{-1}, \mathbf{cbc} = \mathbf{b}^{1+2^n} \rangle, \\
G_{15} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{b}, \mathbf{cbc} = \mathbf{a} \rangle, \\
G_{16} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}, \mathbf{cbc} = \mathbf{b}^{-1} \rangle, \\
G_{17} &= \langle \mathbf{a}, \mathbf{b}, \mathbf{c} \mid (*), \mathbf{cac} = \mathbf{a}^{1+2^n}, \mathbf{cbc} = \mathbf{b}^{-1+2^n} \rangle,
\end{aligned}$$

where  $(*)$  denotes the collection  $\{\mathbf{a}^{2^{n+1}} = \mathbf{b}^{2^{n+1}} = \mathbf{c}^2 = [\mathbf{a}, \mathbf{b}] = 1\}$  of defining relations.

In [3] we proved

**Theorem 2.2.** *Let  $H_i$  and  $G_i$  be finite  $p$ -groups,  $i = 1, \dots, n$ , such that  $H_i$  is good and  $G_i$  fits into an extension  $1 \rightarrow H_i \rightarrow G_i \rightarrow C_p \rightarrow 1$ .*

*Let  $G$  fit into an extension of the form  $1 \rightarrow H \rightarrow G \rightarrow C_p \rightarrow 1$ , with diagonal action of  $C_p$  by conjugation on  $H = H_1 \times \dots \times H_n$ . Denote by*

$$Tr^* = Tr_{\varrho}^* : K(s)^*(BH) \rightarrow K(s)^*(BG),$$

*the transfer homomorphism associated to the  $p$ -covering  $\varrho = \varrho(H, G) : BH \rightarrow BG$ ,*

$$Tr_i^* = Tr_{\varrho_i}^* : K(s)^*(BH_i) \rightarrow K(s)^*(BG_i),$$

*the transfer homomorphism associated to the  $p$ -covering  $\varrho_i = \varrho(H_i, G_i) : BH_i \rightarrow BG_i$ ,  $i = 1, \dots, n$ ,*

$$\rho_i : BG \rightarrow BG_i,$$

*the map, induced by the projection  $H \rightarrow H_i$  on the  $i$ -th factor, and let  $\rho^*$  be the restriction of*

$$(\rho_1, \dots, \rho_n)^* : K(s)^*(BG_1 \times \dots \times BG_n) \rightarrow K(s)^*(BG)$$

*on  $K(s)^*(BG_1)/ImTr_1^* \otimes \dots \otimes K(s)^*(BG_n)/ImTr_n^*$ . Then*

- i) If  $G_i$  are good and so is  $G$ .
- ii)  $K(s)^*(BG)$  is spanned, as a  $K(s)^*(pt)$ -module, by the elements of  $ImTr^*$  and  $Im\rho^*$ .

In particular this implies

**Corollary 2.3.** *Let  $G = G_i$ ,  $i \neq 3, 4, 7, 8, 9, 11$ , then  $G$  is good in the sense of Hopkins-Kuhn-Ravenel.*

*Proof.*  $G_{15}$  is good as wreath product [8]. Otherwise  $G_i$  has maximal abelian subgroup  $H_i = \langle \mathbf{a}, \mathbf{b} \rangle$  on which the quotient acts (diagonally) as above. Each of the following groups  $C_{2^{n+1}} \times C_2$ , the dihedral group  $D_{2^{n+2}}$ , the quasi-dihedral group  $QD_{2^{n+2}}$ , the semi-dihedral group  $SD_{2^{n+2}}$  could be written as semidirect product  $C_{2^{n+1}} \rtimes C_2$  with that kind of action. For all these groups  $K(s)^*(BG)$  is generated by Euler classes, see [16, 17].  $\square$

We will need the following approximations (see [5], Lemma 2.2) for the formal group law in Morava  $K(s)^*$ -theory,  $s > 1$ , we set  $v_s = 1$ .

$$(2) \quad F(x, y) = x + y + (xy)^{2^{s-1}}, \quad \text{mod } (y^{2^{2(s-1)}});$$

$$(3) \quad F(x, y) = x + y + \Phi(x, y)^{2^{s-1}},$$

where  $\Phi(x, y) = xy + (xy)^{2^{s-1}}(x + y) \quad \text{mod } ((xy)^{2^{s-1}}(x + y)^{2^{s-1}}).$

### 3. COMPLEX REPRESENTATIONS OVER $BG$

Let us define some complex representations over  $BG$  we will need.

Let  $H = \langle \mathbf{a}, \mathbf{b} \rangle \cong C_{2^{n+1}} \times C_{2^{n+1}}$  be the maximal abelian subgroup in  $G$ . Let

$$(4) \quad \pi : BH \rightarrow BG$$

be the double covering. Let  $\lambda$  and  $\nu$  denote the following complex line bundles over  $BH$

$$\lambda(\mathbf{a}) = \nu(\mathbf{b}) = e^{2\pi i/2^{n+1}}, \quad \lambda(\mathbf{b}) = \lambda(\mathbf{c}) = \nu(\mathbf{a}) = \nu(\mathbf{c}) = 1,$$

be the pullbacks of the canonical complex line bundles along the projections onto the first and second factor of  $H$  respectively. Let

$$\pi_!(\lambda) = Ind_H^G(\lambda) \quad \text{and} \quad \pi_!(\nu) = Ind_H^G(\nu)$$

be the plane bundles over  $BG$ , the transferred  $\lambda$  and  $\nu$  respectively. Then define three line bundles over  $BG$ ,  $\alpha$ ,  $\beta$  and  $\gamma$  as follows

$$\alpha(\mathbf{a}) = \beta(\mathbf{b}) = \gamma(\mathbf{c}) = -1, \quad \alpha(\mathbf{b}) = \alpha(\mathbf{c}) = \beta(\mathbf{a}) = \beta(\mathbf{c}) = \gamma(\mathbf{a}) = \gamma(\mathbf{b}) = 1.$$

Let us denote Chern classes by

$$x_i = c_i(Ind_H^G(\lambda)), \quad y_i = c_i(Ind_H^G(\nu)), \quad i = 1, 2,$$

$$a = c_1(\alpha), \quad b = c_1(\beta), \quad c = c_1(\gamma).$$

#### 4. PROOF OF THEOREM 1.1

Here we prove that all the remaining groups  $G_i$ ,  $i = 3, 4, 7, 8, 9, 11$ , not covered by Corollary 2.3, are also good.

Our tool shall be the Serre spectral sequence

$$(5) \quad E_2 = H^*(BQ), K(s)^*(BH)) \Rightarrow K(s)^*(BG)$$

associated to a group extension  $1 \rightarrow H \rightarrow G \rightarrow C_2 \rightarrow 1$ . Here  $H^*(BC_2), K(s)^*(BH)$  denotes the ordinary cohomology of  $BC_2$  with coefficients in the  $\mathbb{F}_2[C_2]$ -module  $K(s)^*(BH)$ , where the action of  $C_2$  is induced by conjugation in  $G$ .

Let  $Tr^* : K(s)^*(BH) \rightarrow K(s)^*(BG)$  be the transfer homomorphism [1],[10], [6]. associated to the double covering  $\pi : BH \rightarrow BG$ .

We use the notations of previous two sections. In particular let

$$H \cong C_{2^{n+1}} \times C_{2^{n+1}} \cong \langle \mathbf{a}, \mathbf{b} \rangle.$$

Consider the decomposition

$$(6) \quad [K(s)^*(BH)]^{C_2} = [F]^{C_2} + T,$$

corresponding to the decomposition of  $K(s)^*(BH)$  into free and trivial  $C_2$ -modules. The action of the involution  $t \in C_2$  on

$$(7) \quad K(s)^*(BH) = K(s)^*[u, v]/(u^{2^{(n+1)s}}, v^{2^{(n+1)s}})$$

is induced by conjugation action by  $\mathbf{c}$  on  $H$ . Clearly the composition  $\pi^*Tr^* = 1+t$ , the trace map, is onto  $[F]^{C_2}$ . Therefore it suffices to check that all invariants in  $T$  are also represented by good elements.

For all cases of  $G$  let

$$\begin{aligned} u &= e(\lambda), & v &= e(\nu), \text{ as before,} \\ \bar{x}_1 &= u + t(u) = \pi^*(x_1), & \bar{x}_2 &= ut(u) = \pi^*(x_2), \\ \bar{y}_1 &= v + t(v) = \pi^*(y_1), & \bar{y}_2 &= vt(v) = \pi^*(y_2). \end{aligned}$$

We will need the following

**Lemma 4.1.** *Let  $G$  be one of the groups under consideration and  $t \in C_2 = G/H$  be corresponding involution on  $H$ . Then there is a set of monomials  $\{x^\omega\} = \{\bar{x}_1^i \bar{x}_2^j \bar{y}_1^k \bar{y}_2^l\}$ , such that the set  $\{x^\omega, x^\omega u, x^\omega v, x^\omega uv\}$  is a  $K(s)^*$ -basis in  $K(s)^*(BH)$ . In particular one can choose  $\{x^\omega\}$  as follows*

$$\{x^\omega\} = \begin{cases} \{\bar{x}_2^j \bar{y}_1^k \bar{y}_2^l | j < 2^{ns-1}, k < 2^s, l < 2^{(n+1)s-1}\}, & \text{if } G = G_3, \\ \{\bar{x}_1^i \bar{x}_2^j \bar{y}_1^k \bar{y}_2^l | i, k < 2^s, j, l < 2^{ns-1}\}, & \text{if } G = G_4, G_9, \\ \{\bar{x}_1^i \bar{x}_2^j \bar{y}_1^k \bar{y}_2^l | i, k < 2^{ns}, j, l < 2^{s-1}\}, & \text{if } G = G_7, G_8, G_{11}. \end{cases}$$

*Proof.* If ignore the restrictions, the set  $\{x^\omega, x^\omega u, x^\omega v, x^\omega uv\}$ , generates  $K(s)^*(BH)$ : using  $u^2 = u\bar{x}_1 - \bar{x}_2$  and  $v^2 = v\bar{y}_1 - \bar{y}_2$  any polynomial in  $u, v$  can be written as  $g_0 + g_1u + g_2v + g_3uv$  where  $g_i = g_i(\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2)$  are some polynomials. In particular it follows by induction, that

$$(8) \quad u^{2^m} = u\bar{y}_1^{2^m-1} + \sum_{i=1}^m \bar{y}_1^{2^m-2^i} \bar{y}_2^{2^{i-1}},$$

and similarly for  $v^{2^m}$ .

Now for each case we have to explain the restrictions in  $\{x^\omega\}$ . Then the restricted set  $S = \{x^\omega, x^\omega u, x^\omega v, x^\omega uv\}$  will indeed form a  $K^*(s)$ -basis in  $K^*(s)(BH)$  because of its size  $4^{(n+1)s}$ .

Consider  $G_3$ . For the restrictions on  $l$  and  $k$  we have to take into account (2), (7) and the action of the involution  $t$ . In particular

$$\begin{aligned}
t(\lambda) &= \lambda, \quad t(\nu) = \lambda^{2^n} \nu, \\
t(u) &= u, \\
\Rightarrow \bar{x}_1 &= u + t(u) = 0, \\
\bar{x}_2 &= ut(u) = u^2, \\
t(v) &= F(u^{2^{ns}}, v) = v + u^{2^{ns}} + (vu^{2^{ns}})^{2^{s-1}}, \\
\Rightarrow \bar{y}_2^{2^{(n+1)s-1}} &= 0 \\
\bar{y}_1 &= v + t(v) = u^{2^{ns}} + (vu^{2^{ns}})^{2^{s-1}}, \\
\Rightarrow \bar{y}_1^{2^s} &= 0.
\end{aligned}$$

For the restriction on  $j$ , that is, the decomposition of  $\bar{x}_2^{2^{ns-1}}$  in the suggested basis, note that the formula for  $t(v)$  and (8) for  $m = s - 1$  imply

$$\begin{aligned}
\bar{x}_2^{2^{ns-1}} &= u^{2^{ns}} = \bar{y}_1 + (vu^{2^{ns}})^{2^{s-1}} = \\
&\bar{y}_1 + v^{2^{s-1}} (\bar{y}_1 + (vu^{2^{ns}})^{2^{s-1}})^{2^{s-1}} \\
&\bar{y}_1 + v^{2^{s-1}} \bar{y}_1^{2^{s-1}} = \\
&\bar{y}_1 + \bar{y}_1^{2^{s-1}} (v\bar{y}_1^{2^{s-1}-1} + \sum_{i=1}^{s-1} \bar{y}_1^{2^{s-1}-2^i} \bar{y}_2^{2^{i-1}}) = \\
&\bar{y}_1 + v\bar{y}_1^{2^s-1} + \bar{y}_1^{2^{s-1}} \sum_{i=1}^{s-1} \bar{y}_1^{2^{s-1}-2^i} \bar{y}_2^{2^{i-1}}.
\end{aligned}$$

$G_4$ : The involution acts as follows:  $t(\lambda) = \lambda^{2^n+1}$ ,  $t(\nu) = \lambda^{2^n} \nu^{2^n+1}$ , hence

$$(9) \quad t(u) = F(u, u^{2^{ns}}) = u + u^{2^{ns}} + (uu^{2^{ns}})^{2^{s-1}} \quad \text{by (2),}$$

$$(10) \quad t(v) = F(v, F(v^{2^{ns}}, u^{2^{ns}})) = v + F(v^{2^{ns}}, u^{2^{ns}}) + v^{2^{s-1}} F(v^{2^{ns}}, u^{2^{ns}})^{2^{s-1}}.$$

So that  $\bar{x}_1^{2^s} = \bar{y}_1^{2^s} = 0$ .

For the decomposition of  $\bar{x}_2^{2^{ns-1}}$ , note (9) implies

$$\bar{x}_2^{2^{ns-1}} = (ut(u))^{2^{ns-1}} = u^{2^{ns}}.$$

Then by (9) again

$$\bar{x}_2^{2^{ns-1}} = \bar{x}_1 + (u\bar{x}_2^{2^{ns-1}})^{2^{s-1}} = \bar{x}_1 + (u(\bar{x}_1 + (u\bar{x}_2^{2^{ns-1}})^{2^{s-1}}))^{2^{s-1}} = \bar{x}_1 + u^{2^{s-1}} \bar{x}_1^{2^{s-1}}$$

and apply (8) for  $u^{2^{s-1}}$ .

Similarly for  $\bar{y}_2^{2^{s-1}}$ .

The proof for  $G_9$  is completely analogous as it uses the similar formulas for the action of the involution

$$\begin{aligned} t(\lambda) &= \lambda \nu^{2^n}, \quad t(\nu) = \lambda^{2^n} \nu^{2^n+1}, \\ t(u) &= F(u, v^{2^{n_s}}) = u + v^{2^{n_s}} + (uv^{2^{n_s}})^{2^{s-1}}, \\ t(v) &= F(v, F(u^{2^{n_s}}, v^{2^{n_s}})). \end{aligned}$$

$G_7$ : Let  $\bar{\lambda}$  be the complex conjugate to  $\lambda$  and

$$\bar{u} = [-1]_F(u) = e(\bar{\lambda}), \quad \bar{v} = [-1]_F(v) = e(\bar{\nu}).$$

The involution acts as follows

$$\begin{aligned} t(\lambda) &= \bar{\lambda}, \\ t(\nu) &= \lambda^{2^n} \bar{\nu}, \\ t(u) &= \bar{u} \equiv u + (u\bar{u})^{2^{s-1}} \bmod(1+t), & \text{by (3) as } F(u, \bar{u}) = 0 \\ t(v) &= F(\bar{v}, u^{2^{n_s}}) = \bar{v} + u^{2^{n_s}} + (\bar{v}u^{2^{n_s}})^{2^{s-1}}, & \text{by (2).} \end{aligned}$$

It follows

$$0 = u + \bar{u} \bmod(u\bar{u})^{2^{s-1}} \equiv u + \bar{u} \bmod(u^{2^s})$$

therefore

$$\bar{x}_1^{2^{n_s}} = (u + \bar{u})^{2^{n_s}} = 0, \text{ as } u^{2^{(n+1)s}} = 0.$$

Then as  $u\bar{u} = \bar{x}_2$  is nilpotent we can eliminate  $\bar{x}_2^{2^i} = (u\bar{u})^{2^i}$  for  $i > s-1$  in (3) after finite steps of iteration and write  $\bar{x}_2^{2^{s-1}}$  as a polynomial in  $u + \bar{u} = \bar{x}_1$ . We will not need this polynomial explicitly but only

$$\bar{x}_2^{2^{s-1}} \equiv 0 \bmod(1+t).$$

For  $\bar{y}_1^{2^{n_s}} = 0$  apply the formula for  $t(v)$  and take into account  $v + \bar{v} \equiv 0 \bmod v^{2^s}$ .

For the decomposition of  $\bar{y}_2^{2^{s-1}}$  note we have two formulas for  $F(v, t(v)) = e(\lambda^{2^n}) = u^{2^{n_s}}$ , one is (8) and another is (3). Equating these formulas we have an expression of the form

$$\bar{y}_2^{2^{s-1}} = u\bar{x}_1^{2^{n_s}-1} + P(\bar{y}_1, \bar{y}_2), \text{ for some polynomial } P(\bar{y}_1, \bar{y}_2).$$

Again as  $\bar{y}_2$  is nilpotent we can eliminate  $\bar{y}_2^{2^i}$  for  $i > s-1$  in (3) after finite steps of iteration and write  $\bar{y}_2^{2^{s-1}}$  in the suggested basis. Again we only will need that

$$\bar{y}_2^{2^{s-1}} \equiv u\bar{x}_1^{2^{n_s}-1} \bmod Im(1+t).$$

This completes the proof for  $G_7$ . The proofs for  $G_8$  and  $G_{11}$  is analogous. Let us sketch the necessary information for the interested reader to produce detailed proofs.

$G_8$ : the action of the involution is as follows

$$\begin{aligned} t(\lambda) &= \bar{\lambda}\lambda^{2^n}, \quad t(\nu) = \bar{\nu}\lambda^{2^n}\nu^{2^n}, \\ t(u) &= F(\bar{u}, u^{2^{n_s}}), \\ t(v) &= F(\bar{v}, F(u^{2^{n_s}}, v^{2^{n_s}})). \end{aligned}$$

$G_{11}$ : one has

$$\begin{aligned} t(\lambda) &= \bar{\lambda}\nu^{2^n}, \quad t(\nu) = \bar{\nu}\lambda^{2^n}\nu^{2^n}, \\ t(u) &= F(\bar{u}, v^{2^{n_s}}), \\ t(v) &= F(\bar{v}, F(u^{2^{n_s}}, v^{2^{n_s}})). \end{aligned}$$

For both cases to get  $\bar{x}_1^{2^{n_s}} = 0$  apply formula for  $t(u)$  and  $u + \bar{u} \equiv 0 \pmod{u^{2^s}}$ . Similarly for  $\bar{y}_1^{2^{n_s}} = 0$ . For the decompositions of  $\bar{x}_2^{2^{s-1}}$  and  $\bar{y}_2^{2^{s-1}}$  apply (3) and (8). In particular for  $G_8$  we have by (3)  $\bar{x}_2^{2^{s-1}} \equiv u^{2^{n_s}}$  modulo some  $\bar{x}_1 f(\bar{y}_1, \bar{x}_2) \in \text{Im}(1+t)$ . Therefore  $\bar{x}_2^{2^{n_s-1}} \equiv 0 \pmod{(1+t)}$  and by (8) for  $u$ , we have

$$\bar{x}_2^{2^{s-1}} \equiv u^{2^{n_s}} \equiv \bar{x}_1^{2^{n_s}-1}u + \bar{x}_2^{2^{n_s-1}} \equiv \bar{x}_1^{2^{n_s}-1}u \pmod{(1+t)}.$$

Similarly  $\bar{y}_2^{2^{n_s-1}} \equiv 0 \pmod{(1+t)}$  and we get

$$\bar{x}_2^{2^{s-1}} \equiv F(u^{2^{n_s}}, v^{2^{n_s}}) \equiv \bar{x}_1^{2^{n_s}-1}u + \bar{y}_1^{2^{n_s-1}}v \pmod{(1+t)}.$$

Thus we obtain

$$\begin{aligned} \bar{x}_1^{2^{n_s}} &= \bar{y}_1^{2^{n_s}} = 0, & \text{if } G &= G_7, G_8, G_9, \\ \bar{x}_2^{2^{s-1}} &\equiv 0, \quad \bar{y}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{n_s}-1}u \pmod{(1+t)}, & \text{if } G &= G_7, \\ \bar{x}_2^{2^{s-1}} &\equiv \bar{x}_1^{2^{n_s}-1}u, \quad \bar{y}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{n_s}-1}u + \bar{y}_1^{2^{n_s-1}}v \pmod{(1+t)}, & \text{if } G &= G_8, \\ \bar{x}_2^{2^{s-1}} &\equiv \bar{y}_1^{2^{n_s}-1}v, \quad \bar{y}_2^{2^{s-1}} \equiv \bar{x}_1^{2^{n_s}-1}u + \bar{y}_1^{2^{n_s-1}}v \pmod{(1+t)}, & \text{if } G &= G_{11}. \end{aligned}$$

□

**Lemma 4.2.** *Let  $g = f_0 + f_1u + f_2v + f_3uv \in K(s)^*(BH)$ , where  $f_i = f_i(\bar{x}_1, \bar{y}_1\bar{x}_2, \bar{y}_2)$  are some polynomials written uniquely in the monomials  $x^\omega$  of Lemma 4.1. Then  $g$  is invariant under involution  $t \in G/H$  iff*

$$f_3\bar{x}_1 = f_3\bar{y}_1 = 0; \quad f_1\bar{x}_1 = f_2\bar{y}_1.$$

*Proof.* We have  $g$  is invariant iff  $g \in \text{Ker}(1+t)$ . Then

$$\begin{aligned} g + t(g) &= f_1(u + t(u)) + f_2(v + t(v)) + f_3(uv + t(uv)) = \\ &= f_1\bar{x}_1 + f_2\bar{y}_1 + f_3(\bar{x}_1\bar{y}_1 + \bar{x}_1v + \bar{y}_1u) \end{aligned}$$

and using Lemma 4.1 the result follows.

□



To prove Theorem 1.1 it suffices to see that all invariants are represented by good elements. It is obvious for the elements  $a + t(a) = \pi^* Tr^*(a)$  in free summand  $[F]^{C_2}$  in (6). Therefore one can work modulo  $Im(1+t)$  and check the elements in trivial summand  $T$ . Let us finish the proof of Theorem 1.1 by Propositions 4.3, i). We will turn to Proposition 4.3 ii) later.

**Proposition 4.3.** *Let  $T'$  be spanned by the set*

for  $G_3$ ,

$$\{\bar{x}_2^j \bar{y}_2^l, \bar{x}_2^j \bar{y}_2^l u, \bar{y}_1^{2^s-1} \bar{x}_2^j \bar{y}_2^l v, \bar{y}_1^{2^s-1} \bar{x}_2^j \bar{y}_2^l uv \mid j < 2^{ns-1}, l < 2^{(n+1)s-1}\},$$

for  $G_4, G_9$ ,

$$\{\bar{x}_2^i \bar{y}_2^j, \bar{x}_1^{2^s-1} \bar{x}_2^i \bar{y}_2^j u, \bar{y}_1^{2^s-1} \bar{x}_2^i \bar{y}_2^j v, \bar{x}_1^{2^s-1} \bar{y}_1^{2^s-1} \bar{x}_2^i \bar{y}_2^j uv \mid i, j < 2^{ns-1}\},$$

for  $G_7, G_8, G_{11}$ ,

$$\{\bar{x}_2^i \bar{y}_2^j, \bar{x}_1^{2^{ns}-1} \bar{x}_2^i \bar{y}_2^j u, \bar{y}_1^{2^{ns}-1} \bar{x}_2^i \bar{y}_2^j v, \bar{x}_1^{2^{ns}-1} \bar{y}_1^{2^{ns}-1} \bar{x}_2^i \bar{y}_2^j uv \mid i, j < 2^{s-1}\}.$$

Then

- i) All terms in  $T'$  are represented by good elements and  $T \subset T'$ .
- ii)  $T = T'$ .

Proof of i).

$G_3$ . The basis set of  $T'$  above is suggested by Lemma 4.1 and Lemma 4.2: it is clear that all its terms are invariants. The terms  $\bar{x}_2^j \bar{y}_1^k \bar{y}_2^l \in Im(1+t)$ ,  $k > 0$  are omitted as we work modulo  $1+t$ . Then all the restrictions follow by

$$\bar{y}_1^{2^s} = 0, \bar{x}_1 = 0, \bar{y}_2^{(n+1)s-1} = 0, \bar{x}_2^{2^{ns}-1} \equiv v \bar{y}_1^{2^s-1} \pmod{1+t}.$$

So that  $T \subset T'$ . Let us check that  $T'$  is generated by the images of Euler classes under  $\pi^*$ , where  $\pi$  is the double covering (4).

By definitions

$$\begin{aligned} \pi^*(\alpha) &= \lambda^{2^n}, & \pi^*(\det \pi_!(\nu) \otimes \alpha) &= \nu \lambda^{2^n} \nu \lambda^{2^n} = \nu^2, \\ \pi^*(v') &= v^{2^s}, & \text{where } v' &= e(\det \pi_!(\nu) \otimes \alpha). \end{aligned}$$

Taking into account (8), for  $m = s$ , we get

$$(11) \quad \pi^*(v') = v^{2^s} = v \bar{y}_1^{2^s-1} + \sum_{i=1}^s \bar{y}_1^{2^s-2^i} \bar{y}_2^{2^{i-1}} = \bar{y}_2^{2^s-1} + v \bar{y}_1^{2^s-1} \pmod{1+t}.$$

By definition  $\bar{x}_2 = \pi^*(x_2)$  and  $\bar{y}_2 = \pi^*(y_2)$ . Combining with (11) this implies that all elements of the first and third parts of the basis set of  $T'$  are  $\pi^*$  images of the sums of Euler classes.

For the rest parts of the basis of  $T'$  note, that the bundle  $\lambda$  can be extended to a bundle over  $BG$ , say  $\lambda'$ , represented by  $\lambda'(\mathbf{a}) = e^{2\pi i/2^{n+1}}$ ,  $\lambda'(\mathbf{b}) = \lambda'(\mathbf{c}) = 1$ . So  $\pi^*(e(\lambda')) = u$ . Then note that the second and last parts is obtained by multiplying by  $u$  of the first and third parts respectively. So that we can easily read off all elements as  $\pi^*$  images of the sums of Euler classes.

$G_4$ . Again the basis for  $T'$  is suggested by by Lemma 4.1: we have  $\bar{x}_1^{2^s} = \bar{y}_1^{2^s} = 0$  and  $\bar{x}_2^{2^{ns}-1}$  and  $\bar{y}_2^{2^{ns}-1}$  are decomposable. Then applying (8) we get

$$\begin{aligned}\pi^*(\det(\pi_!\nu) \otimes \alpha) &= \nu^2, \quad \pi^*(e(\det(\pi_!\nu) \otimes \alpha)) = v^{2^s} \equiv v\bar{y}_1^{2^s-1} + \bar{y}_2^{2^s-1} \pmod{1+t}, \\ \pi^*(\det(\pi_!\lambda) \otimes \alpha\beta) &= \lambda^2, \quad \pi^*(e(\det(\pi_!\lambda) \otimes \alpha\beta)) = u^{2^s} \equiv u\bar{x}_1^{2^s-1} + \bar{x}_2^{2^s-1} \pmod{1+t}.\end{aligned}$$

Thus  $G_4$  is good. The proof for  $G_9$  is completely analogous.

$G_7, G_8, G_{11}$ : It is clear that all elements of the basis elements for  $T'$  are invariants and all restrictions are explained by Lemma 4.1. It suffices to check that all elements are represented by images of the sums of Euler classes.

$G_7$ . The bundle  $\lambda^{2^n}$  and  $\nu^{2^n}$  can be extended to the line bundles over  $BG$ , say  $\lambda'$  and  $\nu'$  respectively. So that

$$\pi^*(e(\nu')) = e(\nu^{2^n}) = v^{2^{ns}} \text{ and } \pi^*(e(\lambda')) = e(\lambda^{2^n}) = u^{2^{ns}}.$$

Applying again (8) we get

$$\begin{aligned}\pi^*e(\lambda') &= u^{2^{ns}} = u\bar{x}_1^{2^{ns}-1} + \sum_{i=1}^{ns} \bar{x}_1^{2^{ns}-2^i} \bar{x}_2^{2^i-1} \equiv \\ &u\bar{x}_1^{2^{ns}-1} + \bar{x}_2^{2^{ns}-1} \pmod{1+t} \equiv \\ &u\bar{x}_1^{2^{ns}-1} \pmod{1+t} \text{ by Lemma 4.1}\end{aligned}$$

Similarly, applying Lemma 4.1 we have for  $G_8$

$$\begin{aligned}\pi^*(e(\det(\pi_!\lambda))) &= u^{2^{ns}} \equiv \bar{x}_1^{2^{ns}-1}u \pmod{1+t}, \\ \pi^*(e(\det(\pi_!\nu))) &= F(u^{2^{ns}}, v^{2^{ns}}) \equiv \bar{x}_1^{2^{ns}-1}u + \bar{y}_1^{2^{ns}-1}v \pmod{1+t}\end{aligned}$$

and for  $G_{11}$

$$\begin{aligned}\pi^*(e(\det(\pi_!\lambda))) &= v^{2^{ns}} \equiv \bar{y}_1^{2^{ns}-1}v \pmod{1+t}, \\ \pi^*(e(\det(\pi_!\nu))) &= F(u^{2^{ns}}, v^{2^{ns}}) \equiv \bar{x}_1^{2^{ns}-1}u + \bar{y}_1^{2^{ns}-1}v \pmod{1+t}.\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proposition 4.3 ii) may have an independent interest. Let us sketch the proof.

Using the Euler characteristic formula of [8], Theorem D, one can compute  $K(s)^*$ -Euler characteristic

$$\chi_{2,s}(G) = \text{rank}_{K(s)^*} K(s)^{\text{even}}(BG),$$

for the classifying spaces of the groups in the title. The answer is as follows.

group	$\chi_{2,s}$
$G_1$	$2^{(2n+3)s},$
$G_2, G_4, G_9$	$2^{2(n+1)s-1} - 2^{2ns-1} + 2^{(2n+1)s},$
$G_3, G_{10}$	$3 \cdot 2^{2(n+1)s-1} - 2^{(2n+1)s-1},$
$G_5, G_6, G_7, G_8, G_{11}, G_{12}$	$2^{2(n+1)s-1} - 2^{2s-1} + 2^{3s},$
$G_{13}, G_{16}$	$2^{2(n+1)s-1} - 2^{(n+2)s-1} + 2^{(n+3)s},$
$G_{14}, G_{15}, G_{17}$	$2^{2(n+1)s-1} - 2^{(n+1)s-1} + 2^{(n+2)s}.$

As  $T \subset T'$  it suffices to prove  $\chi_{2,s}(T) = \chi_{2,s}(T')$ . It is easily checked the relation between the size of the trivial summand  $x = \chi_{2,s}(T)$  and  $\chi_{2,s}(G)$  for all groups under consideration

$$(12) \quad (\chi_{2,s}(H) - x) : 2 + 2^s x = \chi_{2,s}(G).$$

Therefore it suffices to see that the number of basis elements of  $T'$  in Lemma 4.3 i)

$T'$	$\chi_{2,s}(T')$
$G_3$	$2^{(2n+1)s},$
$G_4, G_9$	$4^{ns},$
$G_7, G_8, G_{11}$	$4^s.$

is equal to  $x$  in (12) for all cases.

□

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